Assignment 2 [1]

Soft deadline: December 21st, hard deadline (half points): end of the semester

1. What is the matrix F for QFT when $n = 1$ and $n = 2$? Show that F is unitary for any n.

2. Prove that QFT^{-1} (that is, an operator for phase estimation) acts on base vectors as follows:
$$|x\rangle \mapsto \sum_{z=0}^{2^n-1} e^{-2\pi i \frac{x}{2^n} z} |z\rangle$$
(for integer z we mean its binary expansion when writing $|z\rangle$). *Note*: definition of QFT^{-1} was given in terms of action on superpositions with exp-coefficients.

3. When $n = 3$, work out the Quantum Fourier transform of the state $\frac{1}{2}(|000\rangle + |001\rangle + |010\rangle + |011\rangle)$.

4. A quantum algorithm works on two registers R_1 and R_2, initialized to $|0...0\rangle$ and $|u\rangle$ respectively. Suppose that after completion, if the first register is measured in the standard basis, it produces a desired answer $|\varphi_u\rangle$ with probability no less than $(1 - \varepsilon)$. Now suppose that there are several such $|u\rangle$s, with the us elements of a set T, such that all the different $|u\rangle$s are orthogonal to each other, and such that for each u, the probability of producing the desired answer $|\varphi_u\rangle$ is no less than $(1 - \varepsilon)$. The algorithm is now run on the state $|0...0\rangle \otimes \left(\sum_{u \in T} d_u |u\rangle\right)$. Show that the probability that measurement of the first register yields the outcome $|\varphi_u\rangle$ is no less than $|d_u|^2(1 - \varepsilon)$.

5. Prove that if $0 < a < N$, $\gcd(a, N) = 1$, and $L = \lceil \log(N) \rceil$, then the operation U_a on $H^{\otimes L}$ given by
$$U_a|x\rangle = |ax \ (\mathrm{mod} N)\rangle, \quad \text{for } 0 \leqslant x < N,$$
$$U_a|x\rangle = |x\rangle, \quad \text{for } N \leqslant x < 2^L - 1.$$
is a) invertible and b) unitary.

[1] Sources for this text can be found at `https://www.overleaf.com/read/hhmqbgcsnmzv` and used as a reference.