Lecture 8

Continuous random variables

Continuous random variable, PDF

Consider random variables with a continuous range of possible values.

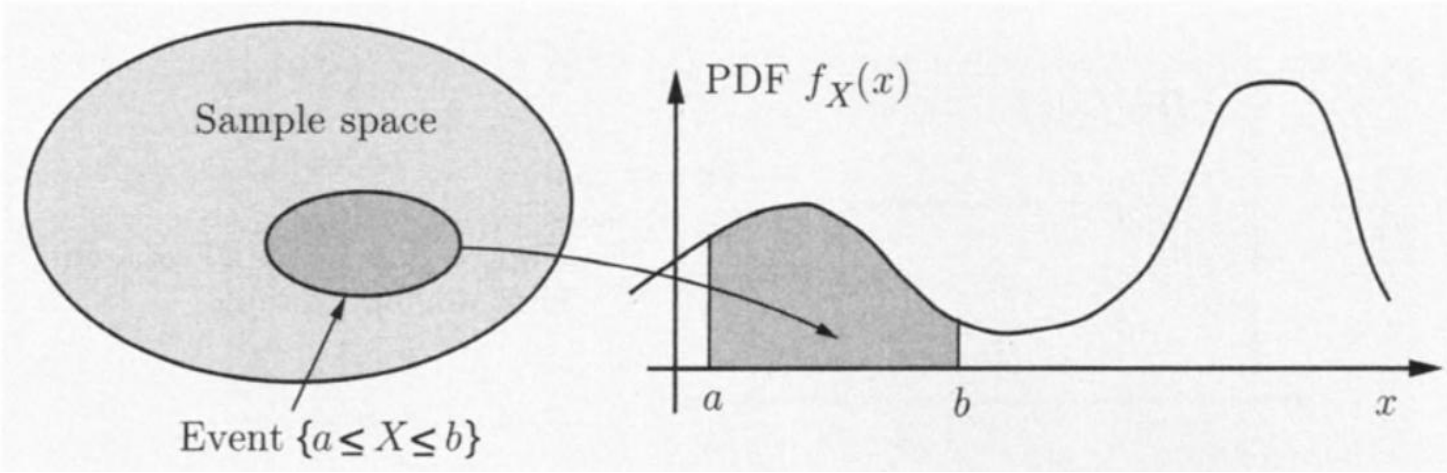
A random variable *X* is called **continuous** if there is a nonnegative function f, called **the probability density function** of *X* ***(PDF)*** such that:

$$P(X \in B) = \int_B f(x)dx$$

for every subset *B* of the real line.

In particular, the probability that the value of *X* falls within an interval is

$$P(a \leq X \leq B) = \int_a^b f(x)dx$$



Properties of the Probability Density Function

Non-Negativity. For every real number x, $f(x) \geq 0$.

Total Area. The total area under the graph of the probability density function is equal to 1:

$$\int_{-\infty}^{+\infty} \boldsymbol{f(x)dx = 1}$$

Probability of an Interval. For any real numbers $a < b$, the probability that X takes a value in the interval $[a, b]$ is equal to the area under the density curve over this interval:

$$P(a \leq X \leq b) = \int_a^b f(x)dx\,.$$

Probability at a Single Point. For every real number x, $P(X = x) = 0$. Therefore, the inclusion or exclusion of the endpoints does not affect the probability: $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$.

Relationship with the Cumulative Distribution Function. The cumulative distribution function of X is

$$\boldsymbol{F(x) = \int_{-\infty}^{x} f(t)dt}$$

At every point where $F(x)$ is differentiable,

$$\boldsymbol{f(x) = F'(x).}$$

Properties of the Probability Density Function

Example. Let *X* be a continuous random variable with the probability density function

$$f(x) = \begin{cases} 2x, 0 \leq x \leq 1 \\ 0, \text{otherwise} \end{cases}$$

Verify that *f(x)* is a valid probability density function.

Cumulative distribution function (CDF)

Let *X* be a continuous random variable. The **cumulative distribution function (CDF)** of X is defined by

$$\boldsymbol{F(x) = P(X \leq x), x \in \mathbb{R}}$$

The cumulative distribution function (CDF) represents the area under the probability density curve to the left of the point *x*.

For any real numbers $a < b$, $\boldsymbol{P(a < X \leq b) = F(b) - F(a).}$

Since a continuous random variable takes any particular value with probability zero, P(X = a) = P(X = b) = 0.

Therefore,

$$\boldsymbol{P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b) = F(b) - F(a).}$$

Cumulative distribution function (CDF)

The cumulative distribution function is obtained by integrating the probability density function:

$$\boldsymbol{F(x) = \int_{-\infty}^{x} f(t)dt}$$

$$\boldsymbol{f(x) = F'(x)}$$

Example. Let X have the probability density function

$$f(x) = \begin{cases} 2x, 0 \leq x \leq 1 \\ 0, \text{otherwise} \end{cases}$$

Find the cumulative distribution function of X.

Continuous random variable, PDF

Expectation: $\boldsymbol{E(X) = \int_{-\infty}^{+\infty} xf(x)dx}$

Variance: $\boldsymbol{var(X) = E(X^2) - (E(X))^2}$**, where** $\boldsymbol{E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx}$

Standard deviation: $\boldsymbol{\sigma = \sqrt{var(X)}}$

Example. Let *X* be a continuous random variable with the probability density function

$$f(x) = \begin{cases} 2x, 0 \leq x \leq 1 \\ 0, \text{otherwise} \end{cases}$$

Cumulative distribution function

Example. The CDF of the random variable X has the form:

$$F(x)=\begin{cases}0, & x<0\\ \frac{1}{33}(2x^2+5x), & 0\le x\le 3\\ 1, & x>3\end{cases}$$

Find PDF; E(X); var(X); $\sigma\ (X), P(1<X<2)$; plot graphs of CDF and PDF.

Example

Example. The PDF of the random variable X has the form:

$$f(x)=\begin{cases}\frac{C}{x^4}, & x\geq 2\\ 0, & x<2\end{cases}$$

Find C; CDF; E(X); var(X); $\sigma\ (X)$; $P(2<X<3)$, plot graphs of CDF and PDF.

Example

Example. The CDF of the random variable T, the service life of the device, is known

$$F(t)=\begin{cases}0, & t<0\\ kt^2, & 0\le t\le \frac{3}{4}\\ 1, & t>\frac{3}{4}\end{cases}$$

Find k, average device life, and variance.

Tasks

Tasks

1. The random variable *X* is given on the entire *Ox* axis by the distribution function $F(x) = \frac{1}{2} + (arctgx)/\pi$. Find the probability that, as a result of the test, the random variable X will take on a value contained in the interval (0.1).

2. The random variable X (uptime of some device) is given on the entire axis by the distribution function $F(x) = 1 - e^{-\frac{x}{T}}\ (x \geq 0)$. Find the probability of failure-free operation of the device over time $x \geq T$.

3. The PDF of a continuous random variable X has the form $f(x) = \frac{2C}{1+x^2}$. Find the constant parameter C.

4. The PDF of the random variable X has the form:

$$f(x) = \begin{cases} 0, & x < 1; \\ A\sqrt{x}, & 1 \leq x \leq 4; \\ 0, & x > 4 \end{cases}$$

Find A; $P(2 < X < 3)$.

Tasks

5. The CDF of the random variable X has the form:

$$F(x)=\begin{cases}0, x\le 0\\ \sqrt{2}\sin x, 0<x\le\dfrac{\pi}{4}\\ 1, x>\dfrac{\pi}{4}\end{cases}$$

Find PDF; E(X); var(X); plot graphs of CDF and PDF.

6. The PDF of the random variable X has the form:

$$f(x)=\begin{cases}A(x-1), & x\in[2;4]\\ 0, & x\notin[2;4]\end{cases}$$

Find A; CDF; plot graphs of CDF and PDF; $P(3<X<7)$; E(X); var(X); $\sigma(X)$.

Tasks

7. The CDF of the random variable X has the form:

$$F(x)=\begin{cases}0, & x<0\\ 1-\cos x, & 0\le x\le \dfrac{\pi}{2}\\ 1, & x>\dfrac{\pi}{2}\end{cases}.$$

Find PDF; E(X); var(X); plot graphs of CDF and PDF.

Tasks

8. Random variable X is given by the CDF:

$$F(x) = \begin{cases} 0, x \leq 0 \\ x^2, 0 < x \leq 1 \\ 1, x > 1 \end{cases}$$

Find PDF; E(X); var(X); $P(0.5 < X < 1)$; plot graphs of CDF and PDF.

9. The CDF of the random variable X has the form:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}x^3, & 0 \leq x \leq 2 \\ 1, & x > 2 \end{cases}.$$

Find PDF; E(X); var(X); $P(1 < X < 2)$; plot graphs of CDF and PDF.

Tasks

10. The cumulative distribution function of a continuous random variable *X* has the form:

$$F(x) = \begin{cases} 0, & x < 0, \\ C(3x^2 - x^3), & 0 \le x \le 2, \\ 1, & x > 2. \end{cases}$$

Find:

- the constant *C*;
- the probability density function *f(x);*
- verify that *f(x)* is a valid probability density function;
- *E(X);*
- *E(X²);*
- *Var(X);*
- *σ(X);*
- *P(0.5 < X < 1.5);*
- plot the graphs of the CDF and PDF.

Homework

Study guide: p. 97-98, Tasks 1-3, 5-7