Lab 3. Symbolic Mathematics

Limits and symbolic differentiation

1. For a sequence defined by a general term , $x_n = \frac{n^2+1}{3n^2-4} + \frac{\sqrt{n}+5}{n+2}$ find its limit at $n \to \infty$.
2. Find the limits of the function $y = \frac{1}{1+2^{1/x}}$ at $x \to +0$ and at $x \to -0$.
3. Find the first and second derivatives of the function $y = x + \sin x$, when the function and its derivative are defined as a) a Maple expression, b) a functional operator. In both cases, calculate the values of the function and its derivative at the point $x = \pi/2$.
4. Find the partial derivative $\frac{\partial^4}{\partial x^3 \partial y}(x^6 \sin(y^3))$ using **`diff`** command and **`D`** operator.
5. Find all first- and second-order partial derivatives of a given function. Simplify the answer if necessary.

$$f(x,y) = \arctan\frac{x+y}{1-xy}.$$

Series expansions

6. Expand in a power series and a Taylor series $f(x) = \arcsin(x)$ in the neighborhood of $x = 0$ and $x = -1$ up to the 8th order. Compare the actions of the two commands. Explain why a Taylor series expansion of the function may not exist. To answer the question, find the derivatives of the function at the given points. Provide your answer in a text comment.
7. Find several partial sums of the Maclaurin series of a function $f = (1-x^2)\arctan(x)$ for n = 3, 5, 10, 20. Plot the graphs of the function and its approximations as partial sums of the Maclaurin series on the interval [-1, 1] on the same axes. Draw the original function as a solid black line, and the graphs of its approximations as dashed lines of different colors. Add a legend with curve captions.

Finding the extrema of a function

8. Find global and local extrema (indicating the nature of the extremum: minimum or maximum) and the extremum points of the functions: $y = \arctan(x) - \frac{\ln(1+x^2)}{2}$, $y = x^4 - x^2$. Use the commands **`extrema`**, **`minimize`**, **`maximize.`** To illustrate your answer, plot the graphs of the functions.
9. Use the **`extrema`**, **`minimize`**, and **`maximize commands`** to find the extrema and extremum points of a function $y = x^3 - 3x^2 + 2$. Plot the function's graph, indicating the coordinates of the extrema in the figure.

Symbolic integration

10. Calculate the indefinite integral $\int \frac{(x^3-6)dx}{x^4+6x^2+8}$.
11. Calculate the definite integrals $\int_1^2 2x^2 dx$, $\int_{-2}^{2} tg(x)dx$, $\int_{-1}^{1} tg(x)dx$. Why might the value of an integral be undefined?
12. Calculate a definite integral $\int_{-1}^{2} \sqrt{x} + \sin(x)dx$. Using the student or **`Student[Calculus 1]`** package commands, calculate the values of the left, right, and middle Riemann sums and plot them

graphically. Determine for what number of approximating rectangles the value of the definite integral and its middle Riemann sum coincide to within 10^{-4} (4 digits after decimal point).

13. Calculate a double integral $\iint_D x - y^2 dxdy$. The integration domain D is bounded by the graphs of the functions $x = 0, x = 1, y = x, y = 2 - x^2$. Calculate the integral in three ways: 1) use **`Doubleint`** command from the **`student`** package, 2) use **`MultiInt`** command from the **`Student[MultivariateCalculus]`** package, 3) use two **`int`** commands (iterated integral). Try to illustrate the graphical meaning of the double integral: graphically depict the integration domain D (here you can use the **`implicitplot`** command) and the surface (defined by the integrand) bounding the volume of the cylinder.
14. Find the area of a plane figure bounded by lines $y = 2x - x^2, y = -x$. To do this plot the graphs of the given functions, calculate the coordinates of the intersection points and use the integral to calculate the area.

Solving algebraic equations

15. Solve a system of equations $\begin{cases} x + 2y + 3z = 8, \\ 3x + y + z = 6, \\ 2x + y + 2z = 6. \end{cases}$ (use **`solve`** command).
16. Find all exact solutions of the system $\begin{cases} x^2 - 5xy + 6y^2 = 0, \\ x^2 + y^2 = 10. \end{cases}$ in analytical form (use **`_EnvExplicit`**)
17. Find all solutions of the trigonometric equation $\sin^4 x - \cos^4 x = 1/2$ (use **`_EnvAllSolutions`**).
18. Find a numerical solution to the equation $e^x = 2(1-x)^2$ (use **`fsolve`** command).
19. Calculate all real roots of the equation $9.5 - x^2 = 4\sin(2x) + 4$ (use the graph to select intervals for the root search and **`fsolve`** command to find the real root). Check each solution by substituting it into the original equation.
20. Plot the graph of the function $y = \sin(x^2) \cdot e^{x/2}$. Identify any two roots in the interval [-3,3], such that the curve between the roots lies above the x-axis and forms a closed region with it. Calculate the area of this region.