Fischer's lemma

Let the vector X consist of independent random variables with a Standard Normal distribution, C is an orthogonal matrix, and

$Y = C \cdot X$.

Then for any k = 1, . . . , n; statistics $T(X) = \sum_{i=1}^{n} X^2{}_i - Y_1{}^2 - \ldots - Y_k{}^2$ does not depend on $Y_1, \ldots, Y_k$ and it has a distribution H_{n-k} with n - k degrees of freedom.

Proof: C – orthogonal matrix, so norm of vectors X and $Y=C \cdot X$ are equal

$$\sum_{i=1}^{n} X_i^2 = \|X\|^2 = \|C \cdot X\|^2 = \sum_{i=1}^{n} Y_i^2$$

so

$$T(X) = \sum_{i=1}^{n} X_i^2 - Y_1^2 - Y_2^2 - \cdots - Y_k^2 = \sum_{i=1}^{n} Y_i^2 - Y_1^2 - Y_2^2 - \cdots - Y_k^2 =$$

$$= Y_{k+1}^2 + Y_2^2 + \cdots + Y_n^2$$

Random variables $Y_1, Y_2, \ldots, Y_n$ - independent and have Normal Standard Distruibution, so

$$T(X) = Y_{k+1}^2 + Y_2^2 + \cdots + Y_n^2$$

have H_{n-k} distribution and doesn't depend on $Y_1, Y_2, \ldots, Y_k$ • *proven* •

a consequence of Fischer's lemma

1) $\sqrt{n}\,\frac{\bar{X}-\mu}{\sigma}$ has Normal Standard distribution for known μ and σ
2) $\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$ has H_n distribution for known μ and σ
3) $\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2 = \frac{(n-1)S_0^2}{\sigma^2}$ has H_{n-1} (Chi-Square) distribution for known σ and unknown μ
4) $\sqrt{n}\,\frac{\bar{X}-\mu}{\sqrt{S_0^2}} = \sqrt{n}\,\frac{\bar{X}-\mu}{S_0}$ has T_{n-1} (Student's) distribution for known μ and unknown σ